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Angular 2-structures

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Abstract

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The theory of 2-structures forms a convenient framework for investigating graphs. In this paper we investigate the class of *angular 2-structures* which results by requiring that the primitivity is forbidden on the lowest possible level, i.e. it is required that no substructure on three elements is primitive. The paper presents the basic theory of angular 2-structures and, in particular, the theory of *primitive* angular 2-structures. We demonstrate that the notion of an angular 2-structure is a well-chosen generalization of the notions of a symmetric graph and a partial order.

Introduction

Among many sorts of graphs considered in the literature, symmetric graphs and partial orders (including linear orders) are certainly important from both theoretical and practical points of view. A *graph* is an ordered pair $g=(D, T)$, where D is a set of *nodes*, and $T \subseteq E_2(D)$ is the set of *edges*. As often done, we assume that g is antireflexive, i.e. $T \subseteq E_2(D)$, where $E_2(D) = D \times D - \{(x,x)\colon x \in D\}$. Then g is *symmetric* iff, for all $(x,y) \in T$, $(y,x) \in T$; g is a *partial order* iff T is transitive; g is a *linear order* iff it is a partial order and, moreover, for each pair $(x,y) \in E_2(D)$, either $(x,y) \in T$ or $(y,x) \in T$.

A natural question to investigate is: is there an "important" common feature of symmetric graphs and partial orders? In this paper we single out such a common feature and argue that it is important.

Consider a graph $g=(D, T)$ and let $(x,y), (u,v) \in E_2(D)$. We say that $(x,y), (u,v)$ are *g-related* iff either $(x,y), (u,v) \in T$, or $(x,y), (u,v) \notin T$. Then we say that g satisfies the *angle* *property* iff, for each set of three nodes $\{x, y, z\}$, either $(x, y), (x, z)$ or $(y, x), (y, z)$ or $(z, x), (z, y)$ are related in g. Now it is easy to see that if g is either a symmetric graph or a partial order, then g satisfies the angle property.

In this paper we investigate the angle property within the theory of 2-structures which forms a convenient framework to investigate graphs (see e.g. [1–3]). A 2-structure satisfying the angle property is called *angular*. In this paper we investigate angular 2-structures. It turns out that we have captured an essential common feature of symmetric graphs and partial orders: we demonstrate that each angular 2-structure can be constructed from symmetric graphs, partial orders and T-structures (which are a natural generalization of linear orders).

The importance of angular 2-structures stems also from the fact that they are well motivated *within* the theory of 2-structures. The key notion of the theory of 2-structures is the notion of primitivity – it is demonstrated in [1–3] that understanding primitivity is crucial for understanding 2-structures. One way to achieve this aim is to forbid the primitivity on the lowest possible level by requiring that no substructure (of a 2-structure) on three elements is primitive. This requirement is equivalent to the requirement of angularity!

0. Preliminaries

We recall now a number of notions concerning sets, relations and graphs, mainly to establish specific notation and terminology for them.

In this paper, unless explicitly stated otherwise, we consider finite sets only. For a set D, $|D|$ denotes its cardinality, and for a family $\mathscr{R}$ of subsets of D we use $\bigcup \mathscr{R}$ to denote $\bigcup_{X \in \mathscr{R}} X$. $\emptyset$ denotes the empty set.

A 2-edge over D is an ordered pair (x, y) such that $x, y \in D$ and $x \neq y$; $E_2(D)$ denotes the set of all 2-edges over D. For a 2-edge (x, y) its *reverse*, denoted by $rev(x, y)$, is the 2-edge (y, x), and its *support*, denoted by $sup(x, y)$, is the set $\{x, y\}$. Given a $P \subseteq E_2(D)$, $sup(P) = \bigcup_{e \in P} sup(e)$.

A *graph* is an ordered pair $h = (D, T)$, where D is a (finite) nonempty set of *nodes*, and $T \subseteq D \times D$ is the set of *edges*. h is *symmetric* iff for each $(x, y) \in T$, $(y, x) \in T$; h is *antisymmetric* iff for each $(x, y) \in T$, $(y, x) \notin T$; h is *antireflexive* (or *loopless*) iff for each $x \in T$, $(x, x) \notin T$. h is called *transitive* iff for all x, y, $z \in D$, $(x, y) \in T$ and $(y, z) \in T$ implies $(x, z) \in T$. h is a *partial order* iff h is antireflexive and transitive. A graph $h' = (D', T')$ is the *reverse* of h iff $D' = D$ and $T' = \{(x, y)\colon (y, x) \in T\}$.

1. 2-structures

In this section we recall (from [1]) the rudiments of the theory of 2-structures.

Definition 1.1. A *2-structure* is an ordered pair (D, R) such that D is a nonempty finite set and R is an equivalence relation on $E_2(D)$.

We use 2s to abbreviate the term "2-structure", and **2S** to denote the class of 2-structures. For a 2s $g=(D,R)$, D is referred to as the *domain of* g, and R as the *equivalence relation of* g; we use $dom(g)$, $rel(g)$, and $2ed(g)$ to denote D, R, $E_2(D)$ respectively. We say that e_1, $e_2 \in E_2(D)$ are *g-equivalent* (or, simply, *equivalent* whenever g is evident from the context) iff $e_1 \, R \, e_2$.

Since for a 2s $g=(D,R)$, R is an equivalence relation on $E_2(D)$, we can specify g in the form $g=(D, \mathscr{P})$, where $\mathscr{P}$ is the partition of $E_2(D)$ induced by R. Depending on the type of consideration, one or the other form of specification may be more convenient; in this paper we use both of them.

From the viewpoint of notation it is convenient to use the following convention. Whenever for a 2s g we write $g=(D,\alpha)$, where α is either $\mathscr{P}$ or $\mathscr{P}$ with a subscript, we mean α to be a partition of $E_2(D)$; on the other hand, if α is either R or R with a subscript, then we mean α to be an equivalence relation on $E_2(D)$.

For a 2s g we use $part(g)$ to denote the partition of $E_2(dom(g))$ induced by $rel(g)$.

Definition 1.2. Let $g=(D,R)$ be a 2s, and let X be a nonempty subset of D. The *substructure of* g *induced by* X, denoted by $sub_g(X)$, is the 2s $(X, R \cap (E_2(X) \times E_2(X)))$. A 2s h is a *substructure of* g iff there exists $X \subseteq D$ such that $h = sub_g(X)$.

Let $g=(D, \mathscr{P})$ be a 2s. Clearly, for each $P \in \mathscr{P}$, (D,P) is a graph and so we can specify g by giving the set of graphs $\mathscr{G}_g = \{(D,P)\colon P \in \mathscr{P}\}$. Since graphs have a convenient pictorial representation, in this way one gets a convenient pictorial representation for g by giving $\mathscr{G}_g$ as one *edge-labeled graph*, where each of the graphs from $\mathscr{G}_g$ gets one edge label, with different graphs getting different edge labels.

Example 1.3. Let $g=(D, \mathscr{P})$, where $D=\{1,2,3,4\}$, and $\mathscr{P}=\{P_1,P_2,P_3,P_4\}$ with

$$P_1=\{(1,2),(3,2),(3,4),(4,3)\},$$

$$P_2=\{(1,3),(2,1),(2,4)\},$$

$$P_3=\{(1,4),(2,3),(4,2)\}, \text{ and}$$

$$P_4=\{(3,1),(4,1)\}.$$

For $X=\{1,3,4\}$, $sub_g(X)=(X, \mathscr{P}')$, where $\mathscr{P}'=\{P'_1,P'_2,P'_3,P'_4\}$ with

$$P'_1=P_1 \cap (X \times X)=\{(3,4),(4,3)\},$$

$$P'_2=P_2 \cap (X \times X)=\{(1,3)\},$$

$$P'_3=P_3 \cap (X \times X)=\{(1,4)\},$$

$$P'_4=P_4 \cap (X \times X)=P_4.$$

The pictorial representation of g (through the pictorial representation of $\mathscr{G}_g$) is as shown in Fig. 1, where 2-edges from different classes of $\mathscr{P}$ get different labels, and all

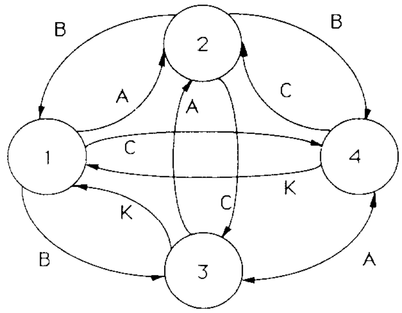

Fig. 1.

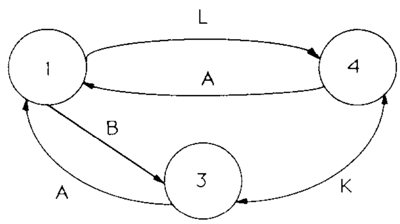

Fig. 2.

2-edges from the same class get the same label (and we use the common pictorial convention of representing an edge from v_1 to v_2 *and* an edge from v_2 to v_1, both labelled by W, by one "symmetric" (double-arrowed) edge between v_1 and v_2 labelled by W.

In the same way $sub_g(X)$ can be represented by the edge-labeled graph shown in Fig. 2.

Remark 1.4. It must be stressed here that in representing $\mathcal{G}_g$ and, hence, g by an edge-labelled graph the choice of labels is *totally arbitrary*. Their only role is to distinguish between different classes of $part(g)$: all 2-edges in one class get the same label, while 2-edges from different classes must get different labels.

This point is well illustrated in the preceding example, where the labeling of our representation of $\mathcal{G}_{sub_g(X)}$ is not related to the labeling of our representation of $\mathcal{G}_g$.

There is a subclass of **2S** that is natural and plays an important role in proving properties of 2-structures. It is defined as follows.

Definition 1.5. A 2s $g=(D,R)$ is *reversible* iff for all e, $e' \in E_2(D)$, $e\,R\,e'$ implies $rev(e)\,R\,rev(e')$.

The condition from the above definition is referred to as the *reversibility condition.* We use r2s to abbreviate the term "reversible 2-structure", and we use **R2S** to denote the class of reversible 2-structures.

Example 1.6. Let $g=(D, \mathscr{P})$, where $D=\{1,2,3,4\}$, and $\mathscr{P}=\{P_1, P_2, P_3, P_4\}$ with

$$P_1=\{(1,2), (2,3)\},$$

$$P_2=\{(2,1), (3,2)\},$$

$$P_3=\{(2,4), (4,2)\}, \text{ and}$$

$$P_4=\{(1,3), (3,1), (1,4), (4,1), (3,4), (4,3)\}.$$

The pictorial representation of g in this case is as shown in Fig. 3. Clearly, g is reversible. On the other hand, the 2s from Example 1.3 is not reversible, e.g. 2-edges $(2,1)$, $(2,4)$ are equivalent, while $(1,2)$, $(4,2)$ are not equivalent.

There is a natural notion of symmetry for 2-edges (and partition classes) of a 2s. It plays an important role in the theory of 2-structures, especially, in the theory of reversible 2-structures.

Definition 1.7. Let $g=(D,R)$ be a 2s.

(1) A 2-edge $e \in E_2(D)$ is *symmetric* (*in* g) iff $e\,R\,rev(e)$; otherwise, e is *asymmetric* (*in* g).

(2) g is called *symmetric* iff all 2-edges of $E_2(D)$ are symmetric; g is *antisymmetric* iff all 2-edges of $E_2(D)$ are asymmetric.

It is easily seen (see [1]) that, for a r2s, g, $e \in 2ed(g)$ is symmetric (asymmetric) iff each e' g-equivalent with e is symmetric (asymmetric). Consequently, each $P \in part(g)$ either consists of symmetric 2-edges only (and then we call P *symmetric*) or it consists of asymmetric 2-edges only (and then we call P *antisymmetric*). Thus, g is symmetric (antisymmetric) iff all $P \in part(g)$ are symmetric (antisymmetric). Clearly, the graph

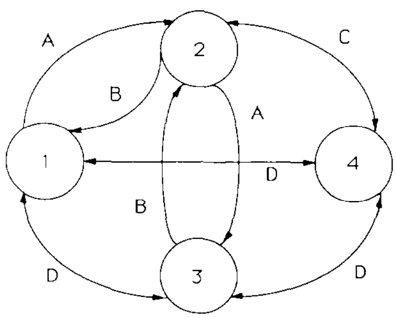

Fig. 3.

(D, P) is symmetric iff P is symmetric, and (D, P) is antisymmetric iff P is antisymmetric; hence, each graph in $\mathcal{G}_g$ is either symmetric or antisymmetric. For $P \in part(g)$ its *reverse*, denoted by $rev(P)$, is the set $\{rev(e)$: $e \in P\}$. It is easily seen (see [1]) that $rev(P) \in part(P)$ and, moreover, P is symmetric iff $rev(P) = P$ and antisymmetric iff $P \cap rev(P) = \emptyset$. Clearly, $(D, rev(P))$ is the reverse graph of (D, P). The *symmetric closure* of P is the set $P \cup rev(P)$; it is denoted by $sym(P)$.

For a r2s g and $P \in part(g)$, the *P-feature* of g is the set $\{P, rev(P)\}$; thus, if P is symmetric, then the P-feature of g is a singleton (and is called *symmetric*); otherwise, it consists of two elements (and is called *antisymmetric*). A *feature of* g is the P-feature of g for a $P \in part(g)$. It is often convenient to specify a r2s g through the set of its features – we write then $g = (D, \mathcal{F})$, where $D = dom(g)$ and $\mathcal{F}$ is the set of all features of g. We also adopt the notational convention that if we write $g = (D, \mathcal{F})$, then we mean $\mathcal{F}$ to be the set of features of g, and F or F with a subscript denotes a feature of g.

Example 1.8. For the r2s g from Example 1.6, P_1, P_2 are antisymmetric, and P_3, P_4 are symmetric; hence, g is neither symmetric nor antisymmetric. Then $P_2 = rev(P_1)$, while $P_3 = rev(P_3)$ and $P_4 = rev(P_4)$. g can be specified through its features: $g = (D, \mathcal{F})$, where $\mathcal{F} = \{F_1, F_2, F_3\}$, with $F_1 = \{P_1, P_2\}$, $F_2 = \{P_3\}$ and $F_3 = \{P_4\}$; F_1 is antisymmetric, while F_2, F_3 are symmetric.

The following notion is the central notion of the theory of 2-structures – it is defined as follows.

Definition 1.9. Let $g = (D, R)$ be a 2s, and let $X \subseteq D$. X is a *clan* (*of* g) iff, for all $x, y \in X$ and all $z \in D - X$, $(z, x)\, R\, (z, y)$ and $(x, z)\, R\, (y, z)$.

Hence, a subset X of the domain of a 2s g is a clan iff all elements of X are "seen in the same way" by each element from outside of X (where different elements outside X may see X in different ways), and each element from outside of X is "seen in the same way" by all elements of X (where different elements outside X may be seen in different ways by elements of X).

We use $\mathcal{C}(g)$ to denote the set of all clans of g. Obviously, $\emptyset \in \mathcal{C}(g)$, $D \in \mathcal{C}(g)$, and $\{x\} \in \mathcal{C}(g)$ for each $x \in dom(g)$; these clans are called *trivial*. We use $\mathcal{TC}(g)$ to denote the set of trivial clans of g and $\mathcal{NTC}(g)$ to denote the set of *nontrivial clans* of g (i.e. the set $\mathcal{C}(g) - \mathcal{TC}(g)$).

Clearly, each 2s g has trivial clans. If g has only trivial clans, then it is *primitive*. Primitive 2-structures are very important in the theory of 2-structures. It is proved in [2] that primitive 2-structures are one of the three kinds of basic blocks from which each 2s can be constructed.

Example 1.10. For the 2s g from Example 1.3, $\mathcal{C}(g) = \mathcal{TC}(g) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3, 4\}\}$; hence, g is primitive.

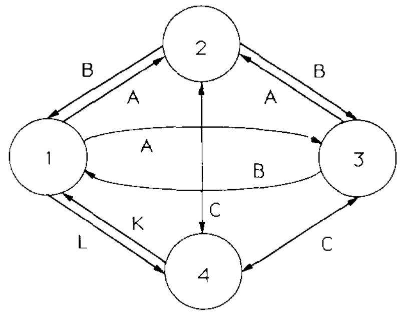

Fig. 4.

Also the 2s from Example 1.6 is primitive.

The r2s shown in Fig. 4 is not primitive, because $\{2,3\}$ is a clan.

The definition of a clan becomes simpler for reversible 2-structures, as stated by Proposition 1.11 (from [1]), following directly from the definition of a clan and from the definition of a r2s.

Proposition 1.11. *Let $g=(D, R)$ be a* r2s *and let $X \subseteq D$. The following statements are equivalent*:

(1) $X \in \mathscr{C}(g)$,

(2) *for all $x, y \in X$ and all $z \in D-X$, $(z,x)\,R\,(z,y)$*,

(3) *for all $x, y \in X$ and all $z \in D-X$, $(x,z)\,R\,(y,z)$*.

The following construction allows one often to consider reversible 2-structures rather than arbitrary 2-structures.

Definition 1.12. (1) Let D be a set, and let $R \subseteq E_2(D) \times E_2(D)$. The *reversible refinement of* R, denoted by $ref(R)$, is a subset of $E_2(D) \times E_2(D)$ defined by

$$\text{for all } e, e' \in E_2(D),\ e\,ref(R)\,e' \text{ iff } e\,R\,e' \text{ and } rev(e)\,R\,rev(e').$$

(2) Let $g=(D,R)$ be a 2s. The *reversible version of* g, denoted by $rver(g)$, is the 2s $(D, ref(R))$.

Example 1.13. (1) For g from Example 1.3, $rver(g)=(D, \mathscr{P}')$, where $D=\{1,2,3,4\}$ and, for each $P \in \mathscr{P}'$, $|P|=1$.

(2) For g from Example 1.6, $rver(g)=g$.

(3) Let g be the 2s shown in Fig. 5. Then $rver(g)$ is the 2s shown in Fig. 6.

The importance of reversible 2-structures in the investigation of 2-structures stems from the following result from [1] (especially, from its last statement).

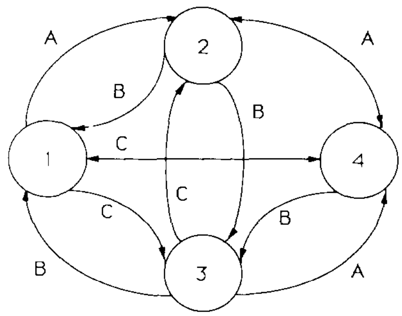

Fig. 5.

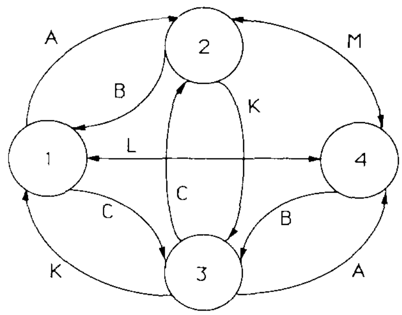

Fig. 6.

Proposition 1.14. *Let g be a* 2s.

(1) *$part(rver(g))$ is a refinement of $part(g)$.*

(2) *$rver(g)$ is reversible.*

(3) *g is reversible iff $g = rver(g)$.*

(4) $\mathscr{C}(g) = \mathscr{C}(rver(g))$.

Hence, $rver(g)$ is indeed reversible and, moreover, the sets of clans of g and $rver(g)$ are equal. On the other hand, the conditions to be satisfied by a set to be a clan of a 2s are simpler for reversible 2-structures (see Proposition 1.11). For this reason, in proving results concerning clans of a 2s g, it is convenient to consider $rver(g)$ rather than g itself – in this sense $rver(g)$ serves as a "normal form" of g.

Remark 1.15. It is easily seen that the following property holds for each 2s g: a 2s h is a substructure of g iff $rver(h)$ is a substructure of $rver(g)$.

It has been proved in [3] that primitivity is "hereditary" for 2-structures in the following sense.

Proposition 1.16. *Let g be a primitive* 2s *such that* $|dom(g)| \geqslant 3$. *Either there exists a primitive substructure g' of g with* $|dom(g')| = |dom(g)| - 1$, *or there exists a primitive substructure g'' of g with* $|dom(g'')| = |dom(g)| - 2$.

Graphs can be translated into 2-structures as follows.

Consider an antireflexive graph $h = (D, T)$. We classify elements of $E_2(D)$ into four classes, C_b, C_e, C_r, and C_n as follows.

$$C_b = \{(x, y)\colon (x, y) \in T \text{ and } (y, x) \in T\},$$

$$C_e = \{(x, y)\colon (x, y) \in T \text{ and } (y, x) \notin T\},$$

$$C_r = \{(x, y)\colon (x, y) \notin T \text{ and } (y, x) \in T\},$$

$$C_n = \{(x, y)\colon (x, y) \notin T \text{ and } (y, x) \notin T\}.$$

Now let $g = (D, R)$, where $R \subseteq E_2(D) \times E_2(D)$ is defined by

for all $e_1, e_2 \in E_2(D)$, $e_1 \, R \, e_2$ iff there exists $x \in \{b, e, r, n\}$ such that $e_1, e_2 \in C_x$.

It is easily seen that R is an equivalence relation on $E_2(D)$ satisfying the reversibility condition, and so g is a r2s; we refer to g as the *(reversible) 2-structure induced by h.*

It is easily seen that if $h = (D, T)$ is an antireflexive graph and g is the 2s induced by h, then also for the following antireflexive graphs (h_1, h_2, h_3) g is the r2s they induce:

$h_1 = (D, (T - C_b) \cup C_n)$,

h_2 equal to the reverse of h,

h_3 equal to the reverse of h_1.

Moreover, it is easily seen that h, h_1, h_2, h_3 are the only graphs that induce g.

Example 1.17. (1) Let h_1 be the antireflexive graph shown in Fig. 7. Then the r2s g_1 shown in Fig. 8 is the r2s induced by h_1.

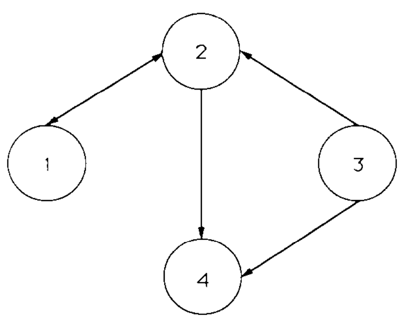

Fig. 7.

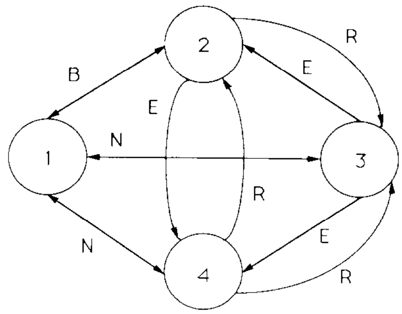

Fig. 8.

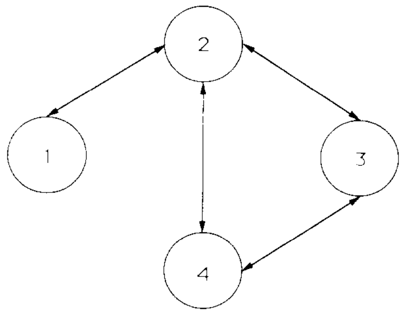

Fig. 9.

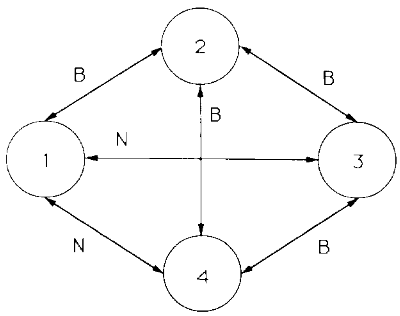

Fig. 10.

(2) Let h_2 be the antireflexive graph shown in Fig. 9. Then the r2s g_2 shown in Fig. 10 is the 2s induced by h_2.

(3) Let h_3 be the antireflexive graph shown in Fig. 11. Then the r2s g_3 shown in Fig. 12 is the 2s induced by h_3, where the labels B, E, R, N are used to label 2-edges from classes C_b, C_e, C_r, and C_n, respectively.

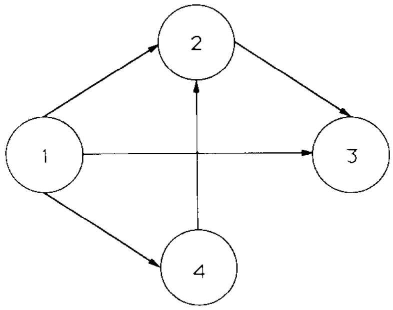

Fig. 11.

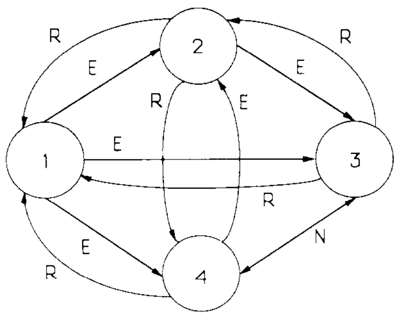

Fig. 12.

Note that in Example 1.17 h_2 is a symmetric graph, and g_2 has two features – both of them symmetric. Also, h_3 is a partial order, and g_3 has two features, one of them symmetric and the other one asymmetric, where both classes P, Q of the asymmetric feature are such that (D_3, P) and (D_3, Q) are partial orders. As a matter of fact this example illustrates the general situation and for this reason we use the following terminology.

A r2s $g=(D, \mathscr{F})$ is a *symmetric graph* iff either $|\mathscr{F}|=1$ or $|\mathscr{F}|=2$ and g is symmetric. g is a *partial order* iff either $|\mathscr{F}|=1$ and g is antisymmetric, or $|\mathscr{F}|=2$, one feature of g being symmetric and the other antisymmetric; moreover, if $F=\{P, Q\}$ is the antisymmetric feature of g, then both (D, P) and (D, Q) are partial orders.

Also, given a r2s $g=(D, \mathscr{P})$ and a $P\in\mathscr{P}$ such that (D, P) is a partial order, we will refer to P as a partial order. Hence, we may refer to a graph, or to a r2s, or to a class of a r2s as a partial order; however, it will always be clear from the context what is meant and, consequently, it should not lead to confusion. Similarly, we will use the term "symmetric graph" referring to a specific type of a reversible 2-structure (as discussed above).

2. Angular 2-structures

In this section angular 2-structures, the topic of this paper, are introduced, and some basic technical properties of angular 2-structures are given.

Definition 2.1. Let $g=(D,R)$ be a 2s.

(1) A triangle Z of g satisfies the *angle property* iff there exists $x \in Z$ such that $(x,y)\,R\,(x,u)$ and $(y,x)\,R\,(u,x)$, where $Z-\{x\}=\{y,u\}$.

(2) g is *angular* iff each triangle of g satisfies the angle property.

We use "A-structure" or simply As to abbreviate the term "angular 2-structure".

Remark 2.2. Angular 2-structures is the subclass of the class of 2-structures where one forbids primitivity on the "lowest possible level". More precisely, let g be a 2s. A subset Z of $dom(g)$ is a *triangle* (*of* g) iff $|Z|=3$. Clearly, g is angular iff, for each triangle Z of g, $sub_g(Z)$ is not primitive.

Angularity is a natural property also from the point of view that (as is easily seen) it is a common feature of symmetric graphs and partial orders. We will prove later that (in a well-defined sense) the notion of an angular 2-structure is a well-chosen generalization of the notions of a symmetric graph and a partial order.

Example 2.3. (1) The 2s from Example 1.3 is not angular – e.g. the triangle $\{1, 2, 3\}$ does not satisfy the angle property. Also the 2s from Example 1.6 is not angular.

(2) The 2s g shown in Fig. 13 is angular.

Since $\mathscr{C}(g)=\mathscr{T}\mathscr{C}(g)$, g is primitive.

Clearly (see Proposition 1.11), a triangle Z of a *reversible* 2s g satisfies the angle property iff there exists $x \in Z$ such that $(x,y)\,R\,(x,u)$, where $Z-\{x\}=\{y,u\}$. Hence, it is technically more convenient to consider *reversible* angular 2-structures (rather than arbitrary angular 2-structures).

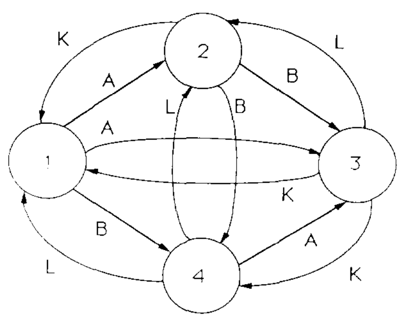

Fig. 13.

For this reason in what follows, up to the end of Section 5, we consider only reversible 2-structures; whenever we write "a 2-structure" we mean "a reversible 2-structure" – in this way our notation becomes simpler. In Section 6 we will discuss how to translate the results of Sections 2 through 5 into the framework of arbitrary (i.e. not necessarily reversible) 2-structures.

The notion of a component "induced" by a 2-edge of a 2s will technically play an important role throughout this paper. It is defined as follows.

Definition 2.4. Let $g=(D,R)$ be a 2s and let $e\in E_2(D)$. The *e-component of g* is the smallest subset Z of D such that $sup(e)\subseteq Z$, and whenever $u\in Z$ and $v\in D$ are such that either $(u,v)\,R\,e$ or $(v,u)\,R\,e$, then $v\in Z$.

For a 2s g and $e\in 2ed(g)$, the e-component of g is denoted by $com_g(e)$. Also, we refer to $com_g(e)$ as "*a component of g*".

The following lemma underlines the investigation of components of an As. To state it we need the following notion. Let $g=(D,\mathscr{P})$ be a 2s, let $x,y\in D$, and let $P\in\mathscr{P}$. We say that x,y are *weakly $\mathscr{P}$-connected* in g iff there exist $x_0,x_1,\ldots,x_n\in D$ such that $n\geqslant 1$, $x_0=x$, $x_n=y$, and, for each $i\in\{1,\ldots,n\}$, $(x_{i-1},x_i)\in sym(P)$.

Lemma 2.5. *Let $g=(D,R)$ be an* As, *and let $e\in 2ed(g)$.*

(1) *Let $P\in part(g)$ be such that $e\in P$. If x,y are different elements of $com_g(e)$, then x,y are weakly P-connected.*

(2) *If $x,y\in com_g(e)$ are such that $(x,y)\,R\,e$, then, for each $z\in D-com_g(e)$, $(z,x)\,R\,(z,y)$.*

Proof. (1) Follows directly from the definition of the e-component of g.

(2) Since $z\notin com_g(e)$, neither $(x,y)\,R\,(x,z)$ nor $(y,x)\,R\,(y,z)$. However, g is angular, and so the triangle $\{x,y,z\}$ satisfies the angle property. Consequently, $(z,x)\,R\,(z,y)$. □

In particular, the above lemma implies that each component of an A-structure is a clan.

Theorem 2.6. *Let g be an* As. *For each $e\in 2ed(g)$, $com_g(e)\in\mathscr{C}(g)$.*

Proof. Directly from Lemma 2.5. □

3. Primitive angular 2-structures

In this section we investigate primitive A-structures. In particular we prove that each primitive As has at most two features; this is the main result of this paper and it will allow us to establish a taxonomy of primitive A-structures. (Recall that in this section, whenever we write "a 2-structure" we mean "a reversible 2-structure").

We begin by the following corollary of Theorem 2.6.

Corollary 3.1. *Let g be a primitive* As.

(1) *For each $e \in 2ed(g)$, $com_g(e) = dom(g)$.*

(2) *For each $x \in dom(g)$ and each $P \in part(g)$ there exists $y \in dom(g)$ such that $(x, y) \in sym(P)$.*

Proof. (1) Follows directly from Theorem 2.6 and the fact that $|com_g(e)| \geqslant 2$.

(2) Assume to the contrary that (2) does not hold; hence, there exist $x_0 \in dom(g)$ and $P_0 \in part(g)$ such that, for each $y \in dom(g)$, $(x_0, y) \notin sym(P_0)$.

Let $e \in P_0$ and consider $com_g(e)$. Then $x_0 \notin com_g(e)$, which contradicts (1). Hence, (2) holds. □

Thus, by Corollary 3.1, each e-component of a primitive 2s g [where $e \in 2ed(g)$] equals the domain of g. Moreover, for each feature F of g, each element $x \in dom(g)$ either has a 2-edge from $\bigcup F$ "incoming to x" or it has a 2-edge from $\bigcup F$ "outgoing from x". In this sense each feature F is "dense" in g.

We prove now that requiring that an As is primitive has an important consequence: the number of features can be at most two! This is an essential difference between A-structures and arbitrary 2-structures: one can have primitive 2-structures with an arbitrary number of features.

Theorem 3.2. *Each primitive* As *has at most 2 features.*

Proof. Let $g = (D, R)$ be a primitive As. We prove the theorem by induction on $|D|$. If $|D| \leqslant 2$, then g has at most one feature. For $|D| = 3$, the theorem trivially holds, because no angular 2s on 3 elements is primitive. If $|D| = 4$, then $|E_2(D)| = 12$; so, it is easily seen that if g has more than 2 features, then [because of Corollary 3.1(2)] g cannot be angular; a contradiction.

Assume now that the theorem holds whenever $|D| \leqslant n$, where $n \geqslant 4$, and consider the case of $|D| = n + 1$.

By Proposition 1.16, either there exists a primitive substructure g' of g such that $|dom(g')| = n$, or there exists a primitive substructure g'' of g such that $|dom(g'')| = n - 1$. We consider these two cases separately.

Case 1: There exists a primitive substructure g' of g such that $|dom(g')| = n$.

Assume to the contrary, that g has at least 3 features; hence, $g = (D, \{F_1, \ldots, F_k\})$ with $k \geqslant 3$.

Let $g' = (D', R')$ be such a primitive substructure and let $D - D' = \{x\}$. By the inductive assumption g' has at most 2 features and, since $|D'| \geqslant 4$ and g' is primitive, g' has precisely 2 features. We may assume that $g' = (D', \{F'_1, F'_2\})$, where F'_1, F'_2 are the restrictions of F_1 and F_2, respectively, to D'.

Since $(\bigcup F_3) \cap 2ed(g') = \emptyset$, by Corollary 3.1(2), for each $y \in D'$, $(y, x) \in \bigcup F_3$. Consequently, if $e \in 2ed(g')$, then $x \notin com_g(e)$, which contradicts Corollary 3.1(1).

Hence, in Case 1 the assumption that g has at least 3 features leads to a contradiction.

Case 2: There exists a primitive substructure g'' of g such that $|dom(g'')| = n-1$.

Let $g'' = (D'', R'')$ be such a primitive substructure and let $D - D'' = \{x_1, x_2\}$. By the inductive assumption, g'' has at most 2 features and, since $|D''| \geq 3$ and g'' is primitive, g'' has precisely 2 features.

(i) Assume to the contrary that g has at least 4 features; hence, $g = (D, \{F_1, \ldots, F_k\})$ with $k \geq 4$. We may assume that $g'' = (D'', \{F_1'', F_2''\})$, where F_1'', F_2'' are the restrictions of F_1 and F_2, respectively, to D''.

Since $(\bigcup(F_3 \cup F_4)) \cap 2ed(g'') = \emptyset$, by Corollary 3.1(2), for each $y \in D''$, $\{x: (y, x) \in \bigcup(F_3 \cup F_4)\} = \{x_1, x_2\}$. Consequently, if $e \in 2ed(g'')$, then $\{x_1, x_2\} \cap com_g(e) = \emptyset$, which contradicts Corollary 3.1(1).

Hence, the assumption that g has at least 4 features leads to a contradiction.

(ii) Assume to the contrary that g has precisely 3 features; hence, $g = (D, \{F_1, F_2, F_3\})$. We may assume that $g'' = (D'', \{F_1'', F_2''\})$, where F_1'', F_2'' are restrictions of F_1 and F_2, respectively, to D''. First of all we note that $(x_1, x_2) \notin \bigcup F_1$. This is seen as follows.

Assume to the contrary that $(x_1, x_2) \in \bigcup F_1$. Then by Corollary 3.1(2), $(x_1, y) \in \bigcup F_2$ for a $y \in D''$. Since $(\bigcup F_3) \cap 2ed(g'') = \emptyset$, by Corollary 3.1(2), $(y, x_2) \in \bigcup F_3$. Then however, the triangle $\{y, x_1, x_2\}$ does not satisfy the angle property, which contradicts the fact that g is angular. Consequently, $(x_1, x_2) \notin \bigcup F_1$. Analogously, we prove that $(x_1, x_2) \notin \bigcup F_2$. Thus, $(x_1, x_2) \notin \bigcup(F_1 \cup F_2)$. Consequently, it must be that $(x_1, x_2) \in \bigcup F_3$. Then, by Corollary 3.1(2), $(x_1, y_1) \in \bigcup F_1$ and $(x_2, y_2) \in \bigcup F_2$ for some $y_1, y_2 \in D''$. Since g is angular, $y_1 \neq y_2$. Since $(\bigcup F_3) \cap 2ed(g'') = \emptyset$, by Corollary 3.1(2), $(y_1, x_2), (y_2, x_1) \in \bigcup F_3$.

Hence, we have the situation as shown in Fig. 14, where the label $i \in \{1, 2, 3\}$ of an edge e indicates that $e \in \bigcup F_i$.

Now, either $(y_1, y_2) \in \bigcup F_1$, or $(y_1, y_2) \in \bigcup F_2$. If $(y_1, y_2) \in \bigcup F_1$, then the triangle $\{y_1, y_2, x_2\}$ does not satisfy the triangle property, and if $(y_1, y_2) \in \bigcup F_2$, then the triangle $\{y_1, y_2, x_1\}$ does not satisfy the angle property. This contradicts the fact that g is angular. Consequently, the assumption that g has precisely 3 features leads to a contradiction.

From (i) and (ii) it follows that in Case 2 the assumption that g has more than 2 features leads to a contradiction. By Proposition 1.16, Cases 1 and 2 imply that g has at most 2 features and, consequently, the theorem holds. □

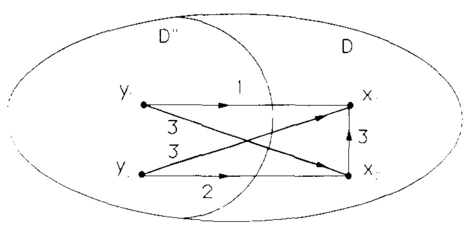

Fig. 14.

The above theorem allows us to classify primitive A-structures into three classes. First we need the following notion.

Definition 3.3. An As g is a T-*structure* iff g is antisymmetric.

Corollary 3.4. *Each primitive* As g *with* $|dom(g)| \geqslant 2$ *is either a symmetric graph, or a partial order, or a* T-*structure.*

Proof. Let g be a primitive As with $|dom(g)| \geqslant 2$. If $|dom(g)| = 2$, then g is either a symmetric graph or a T-structure. Assume now that $|dom(g)| \geqslant 3$. By Theorem 3.2, g has at most 2 features. Since g is primitive and $|dom(g)| \geqslant 3$, g has exactly 2 features. Let $g = (D, \{F_1, F_2\})$. We have then 3 cases to consider.

Case 1: $|F_1| = |F_2| = 1$. Then g has 2 symmetric features and, hence, g is a symmetric graph.

Case 2: One of the features has 2 classes and the other has 1 class; say $|F_1| = 2$ and $|F_2| = 1$. Since g is angular, g is a partial order.

Case 3: $|F_1| = |F_2| = 2$. Then g is antisymmetric, and since g is angular, g is a T-structure.

Remark 3.5. It is proved in [2] that each 2s can be constructed from three sorts of 2-structures: primitive, complete and linear. A 2s $g = (D, \mathscr{F})$ is *complete* iff $|\mathscr{F}| = 1$ and the element of $\mathscr{F}$ is symmetric; g is *linear* iff g is angular, $|\mathscr{F}| = 1$, and the element of $\mathscr{F}$ is antisymmetric. By Corollary 3.4, each primitive As g is either a symmetric graph or a partial order or a T-structure. Clearly, each complete 2s is a symmetric graph, and each linear 2s is a T-structure. Hence, it is easy to see (observing that a substructure of an angular 2s is an angular 2s) that *each* angular 2s can be constructed from symmetric graphs, partial orders and T-structures.

4. Symmetric versions

According to Corollary 3.4, a primitive As is of one of the three possible kinds: a T-structure, a partial order, a symmetric graph. If $g = (D, \mathscr{F})$ is a T-structure with two features (say $\mathscr{F} = \{F_1, F_2\}$) and if one changes one feature of g, say F_1, into a symmetric one by replacing it by $\{\bigcup F_1\}$, then one gets a partial order $g' = (D, \{\bigcup F_1\}, F_2)$. If one does the same transformation again with F_2, then one obtains a symmetric graph $g'' = (D, \{\bigcup F_1\}, \{\bigcup F_2\})$. Hence, by taking a "symmetric version" of a T-structure with two features one gets a partial order, and by taking the "symmetric version" of a partial order one gets a symmetric graph. The question is whether this "symmetric version" operation preserves primitivity, i.e. whether starting with a primitive As g we will get primitive A-structures g' and g''. This question is investigated in this section. (Recall that in this section whenever we write "a 2-structure" we mean "a reversible 2-structure").

Definition 4.1. Let $g=(D,\mathscr{F})$ be a 2s, and let $F\in\mathscr{F}$ be antisymmetric. The *F-symmetric version of g* is the 2s $(D,\mathscr{F}')$, where

$$\mathscr{F}'=(\mathscr{F}-\{F\})\cup\{\bigcup F\}.$$

Since $\bigcup F$ is obviously symmetric, $(D,\mathscr{F}')$ is easily seen to be a 2s and so the F-symmetric version of g is well-defined.

The following lemma is obvious.

Lemma 4.2. *Let $g=(D,\mathscr{F})$ be an* As, *and let $F\in\mathscr{F}$ be antisymmetric. The F-symmetric version of g is also an* As.

We prove now that taking the symmetric version (for an antisymmetric feature of) of a primitive 2s preserves primitivity.

Theorem 4.3. *Let $g=(D,\ \mathscr{F})$ be a primitive* As, *and let $F\in\mathscr{F}$ be antisymmetric. The F-symmetric version of g is also a primitive* As.

Proof. Let $g'=(D,\mathscr{F}')$ be the F-symmetric version of g. By Lemma 4.2, g' is an As. Assume to the contrary that g' is not primitive. Hence, there exists $Z\in\mathscr{N}\mathscr{T}\mathscr{C}(g')$. We define two subsets of $D-Z$:

$$G(Z)=\{x\in D-Z\text{: for all } z_1,\ z_2\in Z,\ (x,\ z_1)\,R\,(x,z_2)\}\text{ and}$$

$$B(Z)=\{x\in D-Z\text{: there exist } z_1,\ z_2\in Z \text{ such that } (x,\ z_1)\,R\,(x,z_2)\text{ does not hold}\},$$

where $R=rel(g)$.

We prove the theorem by the following sequence of observations.

(1) $B(Z)\neq\emptyset$. This follows from the fact that g is primitive and $Z\in\mathscr{N}\mathscr{T}\mathscr{C}(g')$.

(2) For each $y\in B(Z)$ there exist $z_1,z_2\in Z$ such that $(z_1,y)\,R\,(y,z_2)$. This follows from the fact that $Z\in\mathscr{N}\mathscr{T}\mathscr{C}(g')$ while $Z\notin\mathscr{N}\mathscr{T}\mathscr{C}(g)$. For each $y\in B(Z)$ we fix a pair of elements $z_1,z_2\in Z$ satisfying (2) and denote them by $z_1(y)$ and $z_2(y)$, respectively.

(3) $G(Z)\neq\emptyset$. This follows from (1) and Corollary 3.1(2), because, for each $y\in B(Z)$ and each $z\in Z$, $(y,z)\in\bigcup F$.

(4) Let $u\in G(Z)$ and let $P\in part(g)$ be such that $(u,z)\in P$ for each $z\in Z$. Then, for each $y\in B(Z)$, $(u,y)\in P$. To prove this we proceed as follows. Let $y\in B(Z)$. We consider separately two cases.

Case (i): $P\in F$. If $(z_1(y),y)\in P$, then consider the triangle $\{u,z_1(y),y\}$; since g is angular, $(u,y)\in P$. If $(z_1(y),y)\notin P$, then consider the triangle $\{u,z_2(y),y\}$; since g is angular, $(u,y)\in P$.

Case (ii): $P\notin F$. Consider the triangle $\{u,z_1(y),y\}$ and $\{u,z_2(y),y\}$; since g is angular and $(z_1(y),y)$, $(z_2(y),y)$ are not g-equivalent, $(u,y)\in P$.

Now from (3) and (4) it follows that $Z \cup B(Z)$ is a nontrivial clan of g, contradicting the fact that g is primitive. Hence, the assumption that g' is not primitive leads to a contradiction and, consequently, the theorem holds. □

5. Refinements

In this section we consider the "dual" question to the one considered in the previous section. In other words we are interested now in the operation of "splitting" a symmetric class P of an As into antisymmetric and (because of the angularity) transitive classes P_1, P_2. Hence, the operation of getting a partial order from a symmetric graph, and a T-structure from a partial order is considered in this section.

The technical notion underlying the considerations of this section is that of refinement. It is defined as follows. (Recall that in this section whenever we write "a 2-structure", we mean "a reversible 2-structure").

Definition 5.1. Let $g=(D, \mathscr{P})$ be a 2s, and let $P \in \mathscr{P}$ be symmetric.

(1) Let $spl_P \subseteq P \times P$ be the relation defined by

$$\text{for } e_1, e_2 \in P,\ e_1\, spl_P\, e_2 \text{ iff } |sup(e_1) \cap sup(e_2)| = 1 \text{ and } (y_1, y_2) \notin P,$$

where

$$\{y_1, y_2\} = (sup(e_1) \cup sup(e_2)) - (sup(e_1) \cap sup(e_2)).$$

(2) Let $(spl_P)^*$ be the transitive and reflexive closure of spl_P. The partition of P induced by $(spl_P)^*$ is *the refinement of* P.

(3) P is *rudimentary* iff the index of $(spl_P)^*$ equals 1.

Since spl_P is symmetric, $(spl_P)^*$ is an equivalence relation, and so the refinement of P is well defined; we denote it by $ref(P)$.

We prove now that if a class Q of the refinement of a symmetric class has a 2-edge with the support included in a clan Z, then the whole support of Q is included in Z.

Lemma 5.2. *Let $g=(D, \mathscr{P})$ be a 2s, let $P \in \mathscr{P}$ be symmetric, and let $Z \in \mathscr{C}(g)$. If $Q \in ref(P)$ is such that there exists an $e \in Q$ with $sup(e) \subseteq Z$, then $sup(Q) \subseteq Z$.*

Proof. (1) If $x, y \in Z$ and $u \in D$ is such that $(x, y)\, spl_P\, (x, u)$, then $u \in Z$.
This follows, because if $u \in D - Z$, then (u, x), (u, y) are not g-equivalent, which contradicts the fact that $Z \in \mathscr{C}(g)$.

(2) Since $(spl_P)^*$ is the transitive and reflexive closure of spl_P, the lemma follows from (1). □

Next we show that symmetric classes of primitive A-structures do not admit refinements, in the sense that they are rudimentary.

Lemma 5.3. *If $g=(D,\mathscr{P})$ is a primitive* As, *and $P\in\mathscr{P}$ is symmetric, then P is rudimentary.*

Proof. Obviously, the theorem holds if $|D|\leqslant 2$. Since there are no primitive A-structures on three elements, we may assume that $|D|\geqslant 4$. Then, clearly, g has two features, $g=(D,\{F_1, F_2\})$, where $F_2=\{P\}$.

Assume to the contrary that P is not rudimentary; hence, $|ref(P)|\geqslant 2$. Let $Q\in ref(P)$ and let $g'=(D,\{F'_1,F'_2,F'_3\})$, where $F'_1=F_1$, $F'_2=\{Q\}$ and $F'_3=\{P-Q\}$.

(1) *g' is an* As. This is seen as follows. Clearly, g' is a 2s. Consider an arbitrary triangle $\{x,y,z\}$ of g' and assume that it does not satisfy the angle property.

Since $\{x,y,z\}$ satisfies the angle property in g, one of the edges of $\{x,y,z\}$ must be in Q and one in $P-Q$, say, $(x,y)\in Q$ and $(x,z)\in P-Q$. Since $(x,y),(x,z)$ are in different classes of $ref(P)$, and definition of spl_P implies that $(y,z)\in Q\cup(P-Q)$ and, so, either $(y,x),(y,z)\in Q$ or $(z,x),(z,y)\in P-Q$. Hence, $\{x,y,z\}$ satisfies the angle property in g'; a contradiction. Thus, each triangle of g' must satisfy the angle property and, so, g' is an As.

(2) *g' is not primitive.* By (1), g' is an As with three features and, so, by Theorem 3.2, g' is not primitive.

However, if g' is not primitive, then, obviously, g is not primitive; a contradiction. Thus, the assumption that P is not rudimentary leads to a contradiction. Consequently, P must be rudimentary and the lemma holds. □

We prove now that a rudimentary class of a 2s can be "split" into two partial orders in at most one way.

Lemma 5.4. *Let $g=(D,\mathscr{P})$ be a* 2s, *and let $P\in\mathscr{P}$ be symmetric. If P is rudimentary, then there exists at most one partition $\{P_1, P_2\}$ of P such that P_1, P_2 are partial orders.*

Proof. Assume that there exists a partition $\{P_1,P_2\}$ of P, where P_1,P_2 are partial orders. If $|P|=1$, then obviously $\{P_1,P_2\}$ is unique. Hence, assume that $|P|>1$.

(1) Let $(x,y),(x,v)\in P$ be such that $(x,y)\,spl_P(x,v)$, and $(x,y)\in P_1$. Then $(x,v)\in P_1$. This is seen as follows. Assume that $(x,v)\notin P_1$. Then $(v,x)\in P_1$ and, since P_1 is a partial order, the transitivity implies that $(v,y)\in P_1$. But then $(v,y)\in P$, contradicting the fact that $(x,y)\,spl_P(x,v)$. Hence, $(x,v)\in P_1$.

Now take an arbitrary $(x,y)\in P_1$. Since P is rudimentary, (1) implies that (with $(x,y)\in P_1$) each $(u,v)\in P$ is *uniquely* assigned either to P_1 or to P_2. Hence, $\{P_1,P_2\}$ is unique, and the lemma holds. □

We are ready to prove now that there is at most one way of getting from a symmetric graph g to a partial order g' by partitioning a given symmetric class of g into two antisymmetric and transitive classes of g', and there is at most one way of getting from a partial order g' into a T-structure g'' by partitioning the transitive and symmetric class of g' into two antisymmetric and transitive classes of g''.

Theorem 5.5. *Let $g=(D,\mathscr{F})$ be a 2s, and let $g_1=(D,\mathscr{F}_1)$, $g_2=(D,\mathscr{F}_2)$ be primitive A-structures such that there exist antisymmetric $F_1\in\mathscr{F}_1$, $F_2\in\mathscr{F}_2$ for which $\mathscr{F}_1-\{F_1\}=\mathscr{F}_2-\{F_2\}$, g is the F_1-symmetric version of g_1, and g is the F_2-symmetric version of g_2. Then $F_1=F_2$.*

Proof. Since g_1 is a primitive As, and g is the F_1-symmetric version of g_1, by Theorem 4.3, g is a primitive As. Let $F\in\mathscr{F}$ be such that $F=\{P\}$, where $P=\bigcup F_1$. Since P is symmetric, by Lemma 5.3, P is rudimentary. Hence, by Lemma 5.4, $F_1=F_2$. □

Our next result gives a characterization of primitive symmetric graphs. For a 2s g and a $P\in part(g)$, a *P-component of g* is an e-component of g for an $e\in P$.

Theorem 5.6. *A symmetric 2s $g=(D,\{P_1,P_2\})$ is primitive iff*
(1) *each P_1-component of g and each P_2-component of g equals D, and*
(2) *P_1, P_2 are rudimentary.*

Proof. (i) Assume that g is primitive. Since g is symmetric and $|part(g)|=2$, g is an As and, so, a primitive As. Hence, by Corollary 3.1(1), each P_1-component of g and each P_2-component of g equals D; thus, (1) holds, and by Lemma 5.3, both P_1 and P_2 are rudimentary. Hence, (2) holds.

(ii) Assume that conditions (1) and (2) hold. Let $Z\in\mathscr{C}(g)$ be such that $|Z|\geqslant 2$. Consider $(x,y)\in 2ed(g)$ such that $x,y\in Z$. Let $P\in part(g)$ be such that $(x,y)\in P$. By (2), P is rudimentary and, so, by Lemma 5.2, $sup(P)\subseteq Z$. Hence, by (1), $D\subseteq Z$ and, so, $Z=D$.

Thus, for each $Z\in\mathscr{C}(g)$, if $|Z|\geqslant 2$, then $Z=D$. Consequently, g is primitive. □

6. Back to arbitrary (primitive) angular 2-structures

In Sections 2 through 5 we have developed a theory of reversible angular 2-structures and, in particular, a theory of reversible primitive angular 2-structures. What about *arbitrary* (i.e. not necessarily reversible) angular 2-structures? We can easily transfer our results for reversible (primitive) angular 2-structures as follows.

First of all we notice that (by Proposition 1.14 and Remark 1.15):
(1) a 2s g is angular iff $rver(g)$ is angular, and
(2) a 2s g is primitive iff $rver(g)$ is primitive.

In this sense we really use reversibility as a "normal form" for (primitive) angular 2-structures: it is just technically easier to deal with reversible 2-structures.

The main technical result of this paper is Theorem 3.2, which says that a primitive reversible As has at most two features. It yields a taxonomy of primitive reversible angular 2-structures (Corollary 3.1): each primitive reversible As with the domain consisting of at least two elements is either a symmetric graph or a partial order or a T-structure.

We will demonstrate now how to translate Theorem 3.2 (and Corollary 3.1) into arbitrary angular 2-structures. We begin by taking a closer look into the construction of the reversible version of a 2s (see Definition 1.12).

The following lemma follows directly from Definition 1.12.

Lemma 6.1. *Let* $g=(D, \mathscr{P})$ *be a* 2s *and let* $h=rver(g)$, $h=(D, \mathscr{R})$. *Let* $PAIRS(g)=\{(P, P')\in \mathscr{P}\times\mathscr{P}: P\cap rev(P')\neq\emptyset\}$. *Let* ψ *be the mapping from* $\mathscr{R}$ *into* $PAIRS(g)$ *defined by: for each* $R\in\mathscr{R}$, $\psi(R)=(P, P')$, *where* $R\subseteq P$ *and* $rev(R)\subseteq P'$.

(1) *If* g *is not reversible, then* $|\mathscr{R}|>|\mathscr{P}|$.

(2) ψ *is a bijection.*

(3) $R\in\mathscr{R}$ *is symmetric iff* $\psi(R)=(P, P)$ *for some* $P\in\mathscr{P}$.

We will use $pair_g$ to denote the function ψ defined above. To state "the translation" of Theorem 3.2 into arbitrary primitive angular 2-structures we need the following definition.

Definition 6.2. Let $g=(D, \mathscr{P})$ be a 2s.

(1) $P\in\mathscr{P}$ is *antireversible* iff $P\cap rev(P)=\emptyset$.

(2) The *value of* P *(in* g*)*, denoted by $val_g(P)$, is defined by

$$val_g(P)=\begin{cases}1 & \text{if } P \text{ is antireversible,}\\ 2 & \text{otherwise.}\end{cases}$$

(3) The *value of* g, denoted by $val(g)$, is defined by

$$val(g)=\sum_{P\in\mathscr{P}} val_g(P).$$

Theorem 6.3. *For every primitive* As g, $val(g)\leqslant 4$.

Proof. Let $g=(D, \mathscr{P})$ be a primitive As. Obviously, the result holds if $|D|\leqslant 2$. Hence, assume that $|D|\geqslant 3$. If g is reversible, then the theorem follows directly from Corollary 3.1. Hence, assume that g is not reversible. Let $rver(g)=h=(D, \mathscr{R})$. By Corollary 3.1 we have three cases to consider.

Case (1): h is a symmetric graph. Then $\mathscr{R}=\{R_1, R_2\}$, where both R_1 and R_2 are symmetric. Hence, by Lemma 6.1, $pair_g(R_1)=(P_1, P_1)$ and $pair_g(R_2)=(P_2, P_2)$, where P_1, P_2 are symmetric and $\mathscr{P}=\{P_1, P_2\}$. But then $g=h$ and g is reversible; a contradiction.

Hence, h cannot be a symmetric graph.

Case (2): h is a partial order. Then $\mathscr{R}=\{R_{11}, R_{12}, R_2\}$, where $\{R_{11}, R_{12}\}$ is the antisymmetric feature of h and $\{R_2\}$ is the symmetric feature of h. If $|\mathscr{P}|\geqslant 3$, then, by Lemma 6.1, $|\mathscr{R}|>3$; a contradiction. If $|\mathscr{P}|=1$, then g is reversible; a contradiction. Hence, $|\mathscr{P}|=2$, say $P=\{P_1, P_2\}$. Clearly, $\mathscr{P}=\{P_1, P_2\}$, where $P_1\in\{R_{11}, R_{12}\}$ and $P_2=R_2\cup(\{R_{11}, R_{12}\}-P_1)$. Consequently, $val(g)=3$.

Case (3): h is a T-structure. Then $\mathscr{R}=\{R_{11}, R_{12}, R_{21}, R_{22}\}$, where $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$ are antisymmetric features of h. Since g is not reversible, by Lemma 6.1, $|\mathscr{P}|\leqslant 3$. Also by Lemma 6.1, all classes of g must be antireversible and, so, if $|\mathscr{P}|=2$, then $|\mathscr{R}|=2$ and, actually, g is reversible; a contradiction. Hence, $|\mathscr{P}|=3$, where all classes of $\mathscr{P}$ are antireversible. Consequently, $val(g)=3$.

The theorem follows now from Cases 1–3. □

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References

[1] A. Ehrenfeucht and G. Rozenberg, Theory of 2-structures, Part I: Clans, morphisms, and basic subclasses, *Theoret. Comput. Sci.* **70** (1990) 277–304.

[2] A. Ehrenfeucht and G. Rozenberg Theory of 2-structures, Part II: Representation through labeled tree families, *Theoret. Comput. Sci.* **70** (1990) 305–342.

[3] A. Ehrenfeucht and G. Rozenberg, Primitivity is hereditary for 2-structures, *Theoret. Comput. Sci.* **70** (1990) 343–358.